[JULY, 1889.]

THE

TRANSACTIONS

OF THE

ROYAL IRISH ACADEMY.

VOLUME XXIX.—PART XI.

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DUBLIN:
PUBLISHED BY THE ACADEMY,
AT THE ACADEMY HOUSE, 19, DAWSON-STREET.
SOLD ALSO BY
HODGES, FIGGIS, & CO., GRAFTON-STREET.
AND BY WILLIAMS & NORGATE,
LONDON: 14, Henrietta-street, Covent Garden. | EDINBURGH: 20, South Frederick-street.
1889.

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XI.

ON THE MOTION OF A PARTICLE, AND THE EQUILIBRIUM OF FLEXIBLE STRINGS ON A SPHERICAL SURFACE. By T. PRESTON, M.A.

[Read June 24, 1889.]

1. Introductory.—The *Proceedings* of the Royal Irish Academy for July, 1842, contain a communication by the Rev. Charles Graves, F.T.C.D. (now Lord Bishop of Limerick), in which he deduces many of the fundamental equations which determine the motion of a particle on a spherical surface. The system of co-ordinates used by him in arriving at his results, is that which he had already originated, and successfully applied, to the investigation of the properties of conics and higher curves on the sphere. He terms them rectangular spherical co-ordinates. They correspond to the ordinary rectangular Cartesian co-ordinates in plane geometry, and a full account of them will be found in the Appendix to his translation of Chasles' Memoirs on Cones aud Spherical Conics.*

Besides deducing the general equations of motion for central forces, he found, among other interesting results, that a spherical conic will be described under a central force directed towards its focus, and varying inversely as the square of the sine of the angular distance from it; the term "central force" being used here in a sense to be explained immediately.

Any further exposition of these results might therefore seem superfluous; however, as the method of rectangular spherical co-ordinates is not very generally used, and as in this instance it proves somewhat cumbrous, while the method of polar co-ordinates applies with the greatest facility, I may be justified in retaining here, on account of their simplicity, and for the sake of reference, those proofs of the formulæ which I had arrived at before my attention was directed to the communication already mentioned.

* Longmans, 1841.

2. **The Forces.**—If a particle be constrained to move on the surface of a sphere, the forces acting on it at any instant may be resolved into two components—one along the normal to the surface, and another along a tangent line to the sphere. The former does not affect the motion of the particle if the surface be smooth, and it is only the latter, therefore, that claims consideration in the determination of the motion of the particle, as long as it is constrained to move on the surface. This latter, or tangential, component (since at any instant it acts along the element of the great circle to which its direction is a tangent) we may, for definition, term the *great-circle* force, or for brevity, the G. C. F.

If the great circle along which the tangential force acts always passes through a fixed point on the sphere, we may form the analogy *in plano*, say, that particle is acted on by a *Central Force* towards that fixed point, which we may also term the centre of force. It is easily seen that the only limitation thus imposed on the force which really acts on the particle is that its line of action should always meet the diameter of the sphere which passes through the centre of force, as just defined.

For, if a force always meets a given diameter of the sphere, its normal component of course always meets it (since it passes through the centre), and consequently its resultant tangential component must also meet it. That is to say, the great circle which it touches must contain that diameter. The centre of force then is either extremity of that diameter.

A representation of the conditions of motion of a particle, actuated by such a central great-circle force, may be found in the case of a particle tied to a fixed point on the sphere by an elastic string stretched over the surface, and constrained to remain in contact with it; the tension of the string varying as any function of the angular distance of the particle from the point of attachment.

3. **The Energy Equation.**—Taking the radius of the sphere and the mass of the particle throughout to be each unity, we have, by equating the work done by the tangential force to the kinetic energy of the particle,

$$v^2 = 2\int F ds,$$

where ds is an element of the curve described by the particle, and F the

tangential or great-circle force along it. This equation may be written in the form

$$F = v\frac{dv}{ds}.$$

4. **The Equations of Motion under Central Great-circle Forces.**—When the great-circle force always passes through a fixed point on the surface of the sphere, the actual force applied to the particle always meets the diameter through that point. Now, if the force acting on a particle always meets a fixed axis, the moment of momentum round that axis remains constant.

Let O be the fixed point or centre of force; P and P' two consecutive positions of the particle. Then if v be the velocity of the particle along the arc PP' of the sphere which it actually describes, its velocity perpendicular to the axis OC is $v \sin OPP'$; and if we denote OP by r, the perpendicular from P on OC is equal to $\sin r$, taking the radius of the sphere to be unity. Hence the moment of momentum round OC is

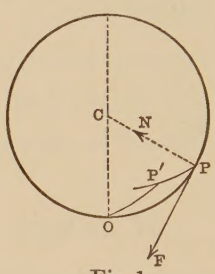

Fig 1.

$$v \sin r \sin OPP'.$$

But
$$\sin r \sin OPP' = \sin p,$$

where p is the perpendicular arc from O on the great circle joining P and P'; that is the great circle tangent arc to the orbit at P.

Hence, the moment of momentum round OC is

$$v \sin p = \text{const.} = h,$$

which we may write in the form

$$hdt = \sin p ds = \sin^2 r d\theta,$$

where $d\theta$ represents the angle POP'; for in the spherical triangle POP', we have

$$\sin p \sin PP' = \sin OP \sin OP' \sin POP'.$$

Consequently

$$\sin^2 r \frac{d\theta}{dt} = h.$$

This equation may also be written down at once from the invariability

of the momentum round OC; for this means that the area described per unit time round O in the tangent plane at O, by the foot of the perpendicular from P on it, remains constant.

The relation $v \sin p = h$ corresponds to that well known *in plano*, viz., $vp = h$, and from it we obtain at once the p and r equation of motion under central forces on the sphere.

For
$$v^2 = \frac{h^2}{\sin^2 p};$$

therefore
$$v\frac{dv}{dr} = -\frac{h^2}{\sin^3 p}\frac{d(\sin p)}{dr},$$

or
$$F = \frac{h^2}{\sin^3 p}\frac{d(\sin p)}{dr}.$$

If we write
$$\cot r = u,$$

we obtain at once the companion expression for F in terms of u and θ. For

$$\frac{1}{\sin^2 p} = \operatorname{cosec}^2 r + \operatorname{cosec}^4 r\left(\frac{dr}{d\theta}\right)^2,$$

since

$$\sin p = \sin^2 r\frac{d\theta}{ds} \quad \text{and} \quad (ds)^2 = (dr)^2 + (\sin r d\theta)^2.$$

Hence

$$\frac{1}{\sin^2 p} = 1 + u^2 + \left(\frac{du}{d\theta}\right)^2,$$

or

$$v^2 = h^2\left\{1 + u^2 + \left(\frac{du}{d\theta}\right)^2\right\}.$$

But
$$v^2 = 2\int F dr = 2\int\frac{F du}{1 + u^2}.$$

Therefore
$$F = h^2\left(1 + u^2\right)\left(u + \frac{d^2u}{d\theta^2}\right).$$

The law of force being known, the orbit is obtained from either of the equations

$$\frac{h^2}{\sin^2 p} = 2\int F dr,$$

or

$$h^2\left\{1 + u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = 2\int F dr = 2\int \frac{F du}{1 + u^2}.$$

EXAMPLE I.

As an example, let us determine the equation of the path of a particle projected along a smooth spherical surface, and acted on by gravity alone.

In this case, the vertical acceleration is g, and the tangential component of this is $g \sin r$, where r is the angular distance of the particle from the lowest point of the sphere. Hence the orbit is described under a central force $g \sin r$ towards the lowest point of the sphere.

The equation of the path is, consequently,

$$\frac{h^2}{\sin^2 p} = 2\int F dr = -2\int g \sin r dr,$$

or

$$\frac{h^2}{\sin^2 p} = 2g \cos r + C,$$

where C is a constant determined by the circumstances of projection.

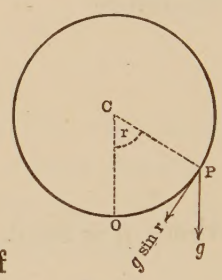

Fig. 2.

Cor. 1.—A particle constrained to move on a spherical surface is attracted towards a fixed point O on the surface by a force directly proportional to the rectilinear distance OP; the orbit is the same as that described under gravity; for the chord OP is equal to $2 \sin \frac{1}{2} r$, and the tangential component of this is $2 \sin \frac{1}{2} r \cos \frac{1}{2} r$, or $\sin r$. The great-circle force towards O is, therefore, $F = \mu \sin r$. Therefore, &c.

Cor. 2.—The apsidal distances are found from the equation

$$\sin^2 r\,(2g \cos r + C) = h^2;$$

for at an apse the radius vector r and the perpendicular p coincide.

Cor. 3.—If the path of the particle be confined to a small portion of the surface around O, that is, if r be always very small, then the path is approximately an ellipse round O as centre; for in this case the force towards O is directly proportional to the distance from it.

Cor. 4.—The determination of the r and θ equation of the orbit leads to an elliptic integral; for

$$\frac{1}{\sin^2 p} = \operatorname{cosec}^2 r + \operatorname{cosec}^4 r \left(\frac{dr}{d\theta}\right)^2.$$

Therefore the equation of the orbit becomes

$$h^2 \left\{ \operatorname{cosec}^2 r + \operatorname{cosec}^4 r \left(\frac{dr}{d\theta}\right)^2 \right\} = 2g \cos r + C,$$

or

$$h \frac{dr}{d\theta} = \sin r \sqrt{\sin^2 r (2g \cos r + C) - h^2}.$$

Therefore

$$\theta = h \int \frac{\sin r dr}{\sin^2 r \sqrt{\sin^2 r (2g \cos r + C) - h^2}}$$

$$= h \int \frac{d\mu}{(\mu^2 - 1) \sqrt{(1 - \mu^2)(2g\mu + C) - h^2}},$$

where $\mu = \cos r$.

EXAMPLE 2.

To find the law of force so that a particle may describe a sphero-conic round the centre of force as a focus.

Let p and p' be the perpendicular arcs from the foci on a great circle tangent to the conic, and let r and r' be the focal radii to the point of contact. Then since the tangent is equally inclined to the focal radii we have

$$\frac{\sin p}{\sin r} = \frac{\sin p'}{\sin r'} = \sqrt{\frac{\sin p \sin p'}{\sin r \sin r'}} = \frac{k}{\sqrt{\sin r \sin r'}},$$

where $k^2 = \sin p \sin p' = b^2/(1 + a^2)$, and a and b are the trigonometrical

tangents of the semiaxes of the conic. Hence

$$\frac{1}{\sin^2 p} = \frac{\sin r'}{k^2 \sin r} = \frac{\sin(2\alpha - r)}{k^2 \sin r} = \frac{1}{k^2}[\sin 2\alpha \cot r - \cos 2\alpha],$$

where α is the semiaxis major of the conic. Consequently

$$F = \tfrac{1}{2}h^2 \frac{d}{dr}\left(\frac{1}{\sin^2 p}\right) = \frac{h^2 \sin 2\alpha}{2k^2} \operatorname{cosec}^2 r = \frac{h^2 a}{b^2} \cdot \frac{1}{\sin^2 r},$$

a and b being, as before, the trigonometrical tangents of the semiaxes of the conic. Hence the force varies inversely as the square of the sine of the angular distance; and if the law of force is given by the equation

$$F = \frac{\mu}{\sin^2 r},$$

we have
$$\mu = \frac{h^2 a}{b^2} \text{ or } \frac{h^2}{\mu} = \frac{b^2}{a}.$$

Cor.—The interpretation of the foregoing law is—*A particle constrained to move on a smooth sphere, and acted on by a force parallel to a given diameter, and varying inversely as the cube of the distance of the particle from that diameter, will describe a sphero-conic having one extremity of the diameter for a focus.*

For the perpendicular distance from P to the diameter is equal to $\sin r$, and consequently the force parallel to the diameter is $\mu / \sin^3 r$, and the tangential component is

$$F = \frac{\mu \sin r}{\sin^3 r} = \frac{\mu}{\sin^2 r}.$$

5. Conversely, the orbit of a particle acted on by a great-circle force towards a fixed point on the sphere is a sphero-conic having the fixed point for a focus, if the force varies inversely as the square of the sine of the angular distance from the fixed point.

Thus if
$$F = \frac{\mu}{\sin^2 r} = \mu \operatorname{cosec}^2 r = \mu(1 + u^2),$$

we have
$$h^2(1+u^2)\left(\frac{du^2}{d\theta^2}\right) = \mu(1+u^2),$$

or
$$u + \frac{d^2u}{d\theta^2} = \frac{\mu}{h^2}.$$

Therefore
$$u - \frac{\mu}{h^2} = A\cos(\theta - \alpha),$$

or
$$\tan r = \frac{h^2}{\mu + Ah^2\cos(\theta - \alpha)}.$$

But the standard polar equation of the spherical conic referred to its focus is
$$\tan r = \frac{b^2/a}{1 \pm \epsilon' \cos\theta}.$$

Consequently the orbit is a spherical conic, having the centre of force for a focus, while the quantities μ and h are related to a and b by the equation $\frac{h^2}{\mu} = \frac{b^2}{a}$.

6. A particle describes a sphero-conic round a centre of force in its focus, to determine the velocity at any point of the orbit. Here
$$v^2 = 2\int F dr = 2\mu\int\frac{dr}{\sin^2 r} = 2\mu\int\sec^2 r\, dr = \frac{2\mu}{\tan r} + C.$$

To determine C consider the particle at the extremity of the axis minor. Here the centrifugal acceleration is equal to $\frac{v_0^2}{\tan\rho_0}$ (see Art. 11), where ρ_0 is the corresponding spherical radius of curvature, and v_0 the corresponding velocity. This must be equal to the component of the applied force along the axis, that is to
$$\frac{\mu}{\sin^2 r_0}\cdot\frac{b}{a},$$

where r_0 is the focal radius to the end of the axis minor and $= \alpha$, the semiaxis

major. Therefore

$$v_0^2 = \frac{\mu \tan \rho_0}{\sin^2 a} = \frac{2\mu}{\tan a} + C.$$

But the radius of curvature at any point of a sphero-conic is given by

$$\tan \rho = \frac{\tan^3 \nu}{p^2},$$

where ν is the intercept made on it by the axis-major, and p the semi-latus rectum, or b^2/a. Hence $\tan \rho_0 = \frac{a^2}{b}$, and consequently

$$\frac{\mu a}{\sin^2 a} = \frac{2\mu}{a} + C.$$

Or, finally,

$$C = \mu\left(a - \frac{1}{a}\right) = \frac{-2\mu}{\tan 2a};$$

therefore

$$v^2 = \frac{2\mu}{\tan r} - \frac{2\mu}{\tan 2a}.$$

Cor.—From this equation it follows that if particles be projected from a fixed point on the surface of a sphere with a given velocity, under a force acting parallel to a given diameter, and varying inversely as the cube of the distance from it, their orbits will be conics, having a common focus at the extremity of the given diameter, and their other foci will lie on a circle having its centre at the point of projection. For v and r being given at the point of projection, it follows that $2a$, the major diameter of the orbit, is given, and therefore r', the angular distance to the other focus, is given.

7. The corollary of the foregoing article leads us to examine the orbit described by a particle acted on by a force parallel to a given diameter of the sphere, and varying as any function of the distance from it.

In this case the force parallel to the diameter is $f(\sin r)$, and the tangential component of this or the great circle force to the extremity of the given diameter is

$$F = \sin r\, f(\sin r).$$

Hence the orbit is

$$\frac{h^2}{\sin^2 p} = 2\int \sin r\, f(\sin r)\, dr;$$

or

$$h^2\left\{1 + u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = 2\int \sin r\, f(\sin r)\, dr + C,$$

where $u = \cot r$.

If the force varies as the nth power of the distance from the diameter, we have the great-circle force

$$F = \mu \sin^{n+1} r,$$

and the orbit is

$$\frac{h^2}{\sin^2 p} = 2\mu \int \sin^{n+1} r\, dr,$$

or

$$h^2\left\{1 + u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = 2\mu \int \sin^{n+1} r\, dr.$$

8. To find the law of force so that a particle may describe a spheroconic round the centre of force as centre.

If p be the perpendicular arc from the centre of the conic on any tangent, r the semi-diameter to the point of contact, and r' the conjugate semi-diameter, we have

$$\tan p \tan r' = ab,$$

where a and b, as before, are the tangents of the same axes of the conic.

Also, since r and r' are conjugate,

$$\tan^2 r + \tan^2 r' = a^2 + b^2.$$

Hence

$$\tan^2 p = \frac{a^2 b^2}{a^2 + b^2 - \tan^2 r};$$

or

$$\cot^2 p = \frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{a^2 b^2}\tan^2 r = 1 + \frac{1}{\sin^2 p},$$

and consequently

$$\frac{1}{\sin^3 p}\frac{d(\sin p)}{dr} = -\frac{1}{a^2 b^2}\tan r \sec^2 r.$$

Therefore,

$$F = \frac{h^2}{a^2 b^2} \frac{\sin r}{\cos^3 r}.$$

In plano this becomes the law of the direct distance.

The interpretation of this equation is, that if the particle be acted on by a force perpendicular to a given diametral plane of the sphere, and inversely as the cube of the distance from it, the orbit will be a sphero-conic, having one pole of the diametral plane for centre.

For if O be the pole of the plane, the perpendicular from P on the plane is equal to $\cos r$, and the force in this direction is $\mu/\cos^3 r$. Hence the tangential component, or the great-circle force towards O, is

$$F = \frac{\mu \sin r}{\cos^3 r}, \quad \text{and} \quad h^2 = \mu a^2 b^2.$$

9. This points to the solution of the more general case when the force is perpendicular to a given diametral plane, and varies as any function of the distance from it (a particle moving under gravity in a spherical bowl being a particular case).

If the force perpendicular to the plane be $\mu f(\cos r)$, its tangential component, or the great-circle force towards the pole of the plane, is

$$F = \mu \sin r f(\cos r).$$

Consequently the equation of the orbit is

$$\frac{h^2}{\sin^2 p} = 2\mu \int \sin r f(\cos r)\, dr = 2\mu\phi(\cos r) + C,$$

or

$$h^2 \left\{1 + u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = 2\mu\phi(\cos r) + C.$$

If the force varies as the nth power of the distance, we have

$$\frac{h^2}{\sin^2 p} = \frac{2\mu \cos^{n+1} r}{n+1} + C.$$

10. Other integrable cases are those in which the force acts perpendicularly to a given diameter, and varies as any function of the distance from it.

Here the force perpendicular to the diameter is $\mu f(\sin r)$, and the great circle force towards the extremity of the diameter is

$$F = \mu \cos r f(\sin r);$$

hence

$$\frac{h^2}{\sin^2 p} = 2\mu \int \cos r f(\sin r)\, dr = 2\mu\phi(\sin r) + C;$$

or

$$h^2 \left\{1 + u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = 2\mu\phi(\sin r) + C.$$

If the force varies as the nth power of the distance, we have

$$F = \mu \cos r \sin^n r,$$

and

$$\frac{h^2}{\sin^2 p} = \frac{2\mu \sin^{n+1} r}{n+1} + C.$$

A particularly interesting case is that in which $n = -3$, for then we have

$$h^2 \left\{1 + u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = -\mu \operatorname{cosec}^2 r + C;$$

or

$$\left(\frac{du}{d\theta}\right)^2 = Au^2 + B,$$

where A and B are constants.

Or, starting from the original equation,

$$F = h^2(1 + u^2)\left(u + \frac{d^2u}{d\theta^2}\right),$$

we have, in this case,

$$F = \mu \cos r \sin^{-3} r = \mu u(1 + u^2);$$

therefore

$$h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \mu u;$$

or

$$\frac{d^2u}{d\theta^2} = -u\left(1 - \frac{\mu}{h}\right).$$

The discussion of the orbit described proceeds now exactly as in the well-known case of *Cotes' Spirals, in plano,* where the law of force is that of the inverse cube of the distance.

Conversely, to find the law of force, so that the curve

$$\cot r = u = Ae^{a\theta} + Be^{-a\theta}$$

may be described, we have

$$F = h^2(1+u^2)\left(u + \frac{d^2u}{d\theta^2}\right) = h^2(1+u^2)u(1+a^2) = h^2(1+a^2)\,\frac{\cos r}{\sin^3 r}.$$

11. **Lemma.**—A particle describes a small circle of a sphere with uniform velocity v: the great circle acceleration towards its spherical centre is $\dfrac{v^2}{\tan\rho}$, where ρ is the angular radius of the circle.

12. We are now in a position to determine in a very simple manner the expressions for the accelerations along, and perpendicular to, the radius vector—the Cartesian methods of deducing these expressions being, even *in plano,* very laborious.

Acceleration along the Radius Vector.—Let O be the origin, and

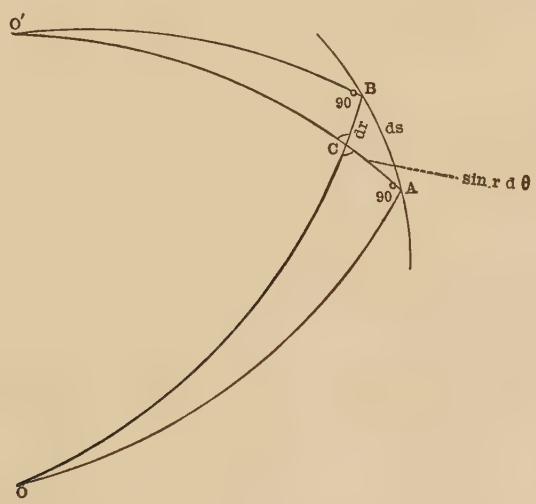

Fig. 3.

$AB = ds$, an element of the curve described by the particle.

The velocity $\frac{ds}{dt}$ of the particle along the curve may be replaced by its components $\frac{dr}{dt}$ and $\sin r \frac{d\theta}{dt}$ along OB and AC, respectively. The velocity $\frac{dr}{dt}$ along OB gives an acceleration $\frac{d^2r}{dt^2}$ along the radius vector, and the velocity $\sin r \frac{d\theta}{dt}$ along AC (which is practically an arc of a small circle* round O as pole) is equivalent to an acceleration towards O, that is, along the radius vector, measured by

$$\frac{\left(\sin r \frac{d\theta}{dt}\right)^2}{\tan r} = \sin r \cos r \left(\frac{d\theta}{dt}\right).$$

Hence the resultant acceleration along the radius vector is

$$\frac{d^2r}{dt^2} - \sin r \cos r \left(\frac{d\theta}{dt}\right)^2.$$

Cor.—In the case of central forces,

$$\sin^2 r \frac{d\theta}{dt} = h,$$

and, by substitution, the acceleration along the radius vector becomes

$$\frac{d^2r}{dt^2} - \frac{h^2 \cos r}{\sin^3 r},$$

from which we infer that if a particle describe an orbit on a sphere under any central force, then the addition of a force acting perpendicularly to a diameter, and varying inversely as the cube of the distance from it, merely alters the value of h in the equation of the orbit. (See Art. 10.)

* If AC is an arc of a small circle round O as pole, then $AC = \sin r d\theta$ accurately; but if AC is an arc of a great circle perpendicular to OA, then $\sin AC = \sin r \sin (d\theta)$, or, to a first approximation, $AC = \sin r d\theta$, which, therefore, amounts to saying that the great circle arc coincides with the small circle arc.

For the equation

$$F = \frac{d^2 r}{dt^2} - \frac{h^2 \cos r}{\sin^3 r}$$

then becomes for the new orbit

$$\frac{\mu \cos r}{\sin^3 r} + F = \frac{d^2 r}{dt^2} - \frac{h'^2 \cos r}{\sin^3 r},$$

or

$$F = \frac{d^2 r}{dt^2} - (h'^2 + \mu) \frac{\cos r}{\sin^3 r},$$

the h of the new orbit being determined from that of the original orbit by the equation

$$h'^2 = h^2 - \mu.$$

This corresponds to that of adding on *in plano* a force varying inversely as the cube of the distance, making the old orbit to revolve,

$$\theta = \kappa\theta', \quad \text{and} \quad h = \kappa h'.$$

13. **Acceleration perpendicular to the Radius Vector.**—As before, th velocity $\sin r \frac{d\theta}{dt}$ along AC gives an acceleration $\frac{d}{dt}\left(\sin r \frac{d\theta}{dt}\right)$ in that direction, while the velocity $\frac{dr}{dt}$ along CB gives an acceleration

$$\frac{\left(\frac{dr}{dt}\right)^2}{\tan r'}, \text{ where } r' = BO',$$

and O' is the point of intersection of the great circles perpendicular to AO and BO, respectively. Hence

$$\tan r' = \tan BCO' \sin BC = \tan BCO' dr,$$

while the triangle AOC gives

$$\tan BCO' = \frac{\tan OA}{\sin AC} = \frac{\tan r}{\sin r d\theta} = \frac{1}{\cos r d\theta}.$$

Therefore

$$\left(\frac{dr}{dt}\right)^2 \div \tan r' = \left(\frac{dr}{dt}\right)^2 \cos r \frac{d\theta}{dr} = \cos r \frac{dr}{dt} \frac{d\theta}{dt},$$

and the complete expression for the acceleration perpendicular to the radius vector is

$$\frac{d}{dt}\left(\sin r\,\frac{d\theta}{dt}\right) + \cos r\,\frac{dr}{dt}\,\frac{d\theta}{dt},$$

or

$$\frac{1}{\sin r}\,\frac{d}{dt}\left(\sin^2 r\,\frac{d\theta}{dt}\right).$$

Cor.—If the force be central, there is no acceleration perpendicular to the radius vector, and, consequently,

$$\sin^2 r\,\frac{d\theta}{dt} = \text{const.} = h,$$

or

$$v \sin p = h.$$

14. A particle P, constrained to move on a spherical surface, is attracted towards a fixed point O on it by a force varying as any function of the rectilinear distance from that point; to find the orbit.

In this case integration is at once possible, for the force along the chord OP may be written in the form $f(\sin\frac{1}{2}r)$, since $OP = 2\sin\frac{1}{2}r$, and the tangential component of this, or the great circle force towards O, is

$$F = \cos\tfrac{1}{2}r\,.f(\sin\tfrac{1}{2}r).$$

Therefore,

$$\frac{h^2}{\sin^2 p} = 2\int \cos\tfrac{1}{2}r f(\sin\tfrac{1}{2}r)\,dr = 4\phi(\sin\tfrac{1}{2}r) + C;$$

or

$$h^2\left\{1 + u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = 4\,\phi\,(\sin\tfrac{1}{2}r) + C.$$

If the force varies as the nth power of the distance, we have

$$F = \mu \sin^n \tfrac{1}{2}r \cos\tfrac{1}{2}r,$$

and

$$\frac{h^2}{\sin^2 p} = \frac{4\mu \sin^{n+1}\frac{1}{2}r}{n+1} + C.$$

For the law of nature $n = -2$, and the orbit is

$$\frac{h^2}{\sin^2 p} = \frac{4\mu}{\sin \frac{1}{2} r} + C.$$

Hence two material particles attracting each other on a sphere describe similar orbits round each other.

15. To express the great-circle force F in terms of the velocity v, and the spherical semi-chord of curvature γ of the orbit.

The orbit being described under a central force, we have

$$v = \frac{h}{\sin p}.$$

And further, the radius of curvature ρ at any point of a spherical curve is given by the equation

$$\tan \rho = \frac{\sin r}{\cos p} \frac{dr}{dp}, \qquad \text{(Williamson's } \textit{Calculus.}\text{)}$$

where r is the radius vector to the point, and p the perpendicular on the tangent at it.

But
$$F = \frac{h^2}{\sin^3 p} \cos p \frac{dp}{dr} = \frac{h^2 \sin r}{\sin^3 p \tan \rho} = \frac{v^2 \sin r}{\sin p \tan \rho}.$$

Now, if γ be the semi-chord of curvature, we have obviously

$$\frac{\sin p}{\sin r} = \frac{\tan \gamma}{\tan \rho}.$$

Therefore
$$F = \frac{v^2}{\tan \gamma},$$

which corresponds to Newton's formula *in plano*, viz. $F = v^2/\gamma$.

Cor. 1.—If the velocity at each point (r, θ) of an orbit is equal to that in a circle of angular radius r, the orbit is an equiangular spiral for an attractive force.

For the velocity in the orbit is $\sqrt{F \tan \gamma}$, and the velocity in the circle is $\sqrt{F \tan r}$.

Hence
$$\tan r = \tan \gamma = \tan \rho \frac{\sin p}{\sin r} = \tan p \frac{dr}{dp};$$

or
$$\frac{dr}{\tan r} = \frac{dp}{\tan p};$$

or
$$\frac{\sin p}{\sin r} = \text{const.}$$

That is, the orbit is such that the angle between the tangent to it at any point, and the radius vector to that point, is constant.

If the force be repulsive, we have

$$\sin p \sin r = \text{const.}$$

In plano this represents an equilateral hyperbola; but on the sphere that curve is represented by the equation

$$\tan p \tan r = \text{const.}$$

From the above equation, we have

$$v^2 = \frac{h^2}{\sin^2 p} = \frac{\mu}{\sin^2 r};$$

or
$$F = v \frac{dv}{dr} = \frac{\mu \cos r}{\sin^3 r} = \mu u (1 + u^2). \quad [\textit{Cf.}\ \text{Art. 10.}]$$

16. In general, if the force in the orbit be in a constant ratio $(1 : \sqrt{n})$ to the force in the circle, we have

$$\sin p = k \sin^n r.$$

Hence
$$v^2 = \frac{\mu}{\sin^{2n} r};$$

or
$$F = v \frac{dv}{dr} = \frac{n\mu \cos r}{\sin^{2n+1} r},$$

which represents a force acting perpendicularly to a given diameter, and varying inversely as the $(2n+1)^{th}$ *power of the distance from it.* [*Cf.* Art. 8.]

Cor. 2.—As another example of the use of the equation $F = v^2/\tan\gamma$, we find at once the analogue of Prof. Curtis's theorem (see Williamson's and Tarleton's *Dynamics*, 2nd ed., p. 224).

If a curve be described freely under the joint action of forces F_1, F_2, &c., acting along r_1, r_2, &c., respectively, and if ϕ_1, ϕ_2, &c., be the forces acting along r_1, r_2, &c., under which, acting singly, the orbit would be described freely, we have

$$v^2 = \Sigma F \tan\gamma;$$

also

$$-vdv = \Sigma Fdr.$$

Hence

$$2\Sigma Fdr + \Sigma\{Fd(\tan\gamma) + \tan\gamma\, dF\} = 0;$$

or

$$\Sigma[F\{d(\tan\gamma) + 2dr\} + \tan\gamma\, dF] = 0.$$

But since the orbit is described freely under any one of the forces ϕ, we have

$$\phi\{d(\tan\gamma) + 2dr\} + \tan\gamma\, d\phi = 0.$$

Therefore, by substitution, we find

$$\Sigma\tan\gamma\left(dF - \frac{Fd\phi}{\phi}\right) = 0;$$

or

$$\Sigma\phi\tan\gamma d\left(\frac{F}{\phi}\right) = 0.$$

CONSTRAINED MOTION.

17. If a particle move on the surface of a smooth sphere with a velocity v, the reaction on the sphere arising from the motion is v^2/a, where a is the radius of the sphere. But if the outward normal component of the forces acting on the particle be $\overline{N}$, then the total normal reaction on the surface will be

$$v^2 + \overline{N},$$

if we take the radius of the sphere and the mass of the particle as unity. This equation enables us to tell at what point of its motion the reaction will change sign, and here the particle will leave the surface if free to do so. We may suppose it constrained to remain on the surface by being confined between two very near concentric spherical shells.

Supposing it to be constrained to always remain on the surface, we may examine also the conditions of its motion when it is further constrained to describe a given curve on the surface, when it is acted on by any given forces.

Thus, if N be the outward component of the great circle-force F, normal to the spherical curve which the particle is constrained to describe, we have the reaction on the curve equal to

$$R = \frac{v^2}{\tan \rho} + N.$$

Hence, if N varies inversely as $\tan \rho$, the particle may be made to describe the orbit freely.

So again, it follows that if a particle describes a spherical curve freely under different forces acting separately, it may be so projected as to describe the same curve when all the forces act conjointly, which is the analogue of the theorem of M. Ossian Bonnet (Williamson's and Tarleton's *Dynamics*, 2nd ed., p. 208).

EXAMPLE.

As an example of the constrained motion of a particle, let us determine the reaction on the curve when a particle describes a spherical conic under great-cicle forces towards the foci, varying inversely as the square of the sine of the angular distance, the force towards one focus being attractive, and that towards the other repulsive, while both are of the same absolute intensity.

In this case we have

$$R = \sin\psi\left[\frac{\mu}{\sin^2 r} - \frac{\mu}{\sin^2 r'}\right] + \frac{v^2}{\tan\rho},$$

where ψ is the angle between the radius vector and the tangent, and consequently

$$\sin\psi = \frac{\sin p}{\sin r} = \frac{\sin p'}{\sin r'} = \sqrt{\frac{\sin p \sin p'}{\sin r \sin r'}},$$

and

$$\sin p \sin p' = \text{const.} = \frac{b^2}{1+a^2},$$

where a and b are the trigonometrical tangents of the semi-axes of the conic.

But at any point we have

$$\tan\rho = \frac{\tan^3\nu}{p^2}$$

[see Graves's Appendix to Chasles' "Memoirs"], where ν is the intercept made on the normal by the major axis of the conic, and p is equal to b^2/a, or what corresponds to the semi-latus rectum. Again, from the expressions for $\sin r$, and $\sin r'$ (Graves's Appendix to Chasles' "Memoirs," pp. 105 and 107), we find at once

$$\sin r \sin r' = \frac{a^2 - \epsilon^2 x'^2}{(1+a^2\epsilon^2)(1+x'^2+y'^2)}$$

$$= \frac{a^4 b^4 + a^4 y'^2 + b^4 x'^2}{a^2 b^2 (1+b^2)(1+a^2\epsilon^2)(1+x'^2+y'^2)}$$

$$= \frac{a^2 \tan^2\nu}{b^2(1+a^2)}.$$

Consequently, since

$$v^2 = 2\mu \int \left(\frac{dr}{\sin^2 r} - \frac{dr'}{\sin^2 r'} \right)$$

$$= 2\mu \left(\cot r - \cot r' \right) + \text{const.},$$

the constant is obviously equal to the velocity at the extremity of the axis minor. Denoting this velocity by v_0, and as usual $\cot r = u$, we have the reaction

$$R = \mu \sin \psi (u^2 - u'^2) - \frac{2\mu(u - u')}{\tan \rho} + \frac{v_0^2}{\tan \rho}$$

$$= \mu(u - u') \left\{ \frac{b(u + u')}{\sqrt{(1 + a^2)} \sin r \sin r'} - \frac{2}{\tan \rho} \right\} + \frac{v_0^2}{\tan \rho}.$$

Now, $$u + u' = \frac{\sin (r + r')}{\sin r \sin r'} = \frac{\sin 2\alpha}{\sin r \sin r'}.$$

Consequently,

$$\frac{b(u + u)}{\sqrt{(1 + a^2)} \sin r \sin r'} = \frac{b \sin 2\alpha}{\sqrt{1 + a^2}} \times \left\{ \frac{b^2(1 + a^2)}{a^2 \tan^2 \nu} \right\}^{\frac{3}{2}}$$

$$= \frac{2b^4}{a^2 \tan^3 \nu} = \frac{2}{\tan \rho}.$$

We have, therefore,

$$R = \frac{v_0^2}{\tan \rho},$$

or the reaction varies inversely as the tangent of the radius of curvature. Hence, if the velocity at the extremity of the axis minor be zero, the particle will describe the semi-ellipse freely; that is, if it be placed at rest at the end of the axis minor, it will oscillate through one-half of the ellipse.

EQUILIBRIUM OF STRINGS.

18. Let AB be an element of a flexible string placed on a sphere, and in equilibrium under a central great-circle force emanating from a point O on the surface of the sphere. Since the only condition imposed on the forces acting on the string by their being *central* is, that they shall always meet the diameter through O, it follows that the moments round this diameter of the tensions at the extremities of the element are equal and opposite. But if T be the tension at A, its moment round OC is

$$T \sin \epsilon \sin r,$$

where $r = OA$, and ϵ is the angle between OA and AB (the perpendicular from A on OC being $\sin r$).

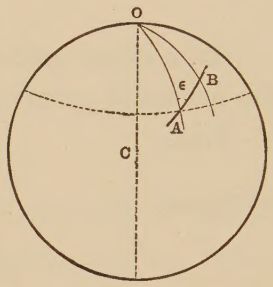

Fig. 4.

But $$\sin \epsilon \sin r = \sin p,$$

where p is the perpendicular arc from O on the great circle AB, which is the great circle tangent at A to the curve of the string.

Hence, the moment of the tension at A is $T \sin p$, and, consequently, for equilibrium we have

$$T \sin p = \text{const.} = h. \tag{1}$$

Again, by resolving the forces in the direction of the element AB, we have

$$-F \frac{dr}{ds} . mds = dT,$$

m being the mass of the string per unit length; therefore

$$dT = -mFdr, \tag{2}$$

the negative sign applying to repulsive forces. These equations (1) and (2) determine the form of the curve of equilibrium of the string.

Eliminating T, we have

$$\frac{h}{\sin p} = -\int mFdr = \int \frac{Fdu}{1+u^2} = \phi(u),$$

where $u = \cot r$,

or
$$h^2\left\{1 + u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = \left\{\phi(u)\right\}^2,$$

or
$$h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \phi(u)\,\phi'(u).*$$

EXAMPLE.

An endless uniform flexible string is placed on a smooth sphere under the action of gravity; find the curve of equilibrium.

Here
$$F = g \sin r;$$
consequently,
$$\frac{h}{\sin p} = -\int g \sin r dr = g \cos r + C, \text{ \&c.}$$

The subject may be pursued in other cases of central forces similarly to those already treated of in the motion of a particle.

* See a Paper by Professor Townsend, in the *Quarterly Journal of Pure and Applied Mathematics*, 1874, or Minchin's *Statics*.